

L^∞ -UNIQUENESS OF GENERALIZED SCHRÖDINGER OPERATORS

Ludovic Dan LEMLE*

version 19 December 2007

Abstract

The main purpose of this paper is to show that the generalized Schrödinger operator $\mathcal{A}^V f = \frac{1}{2}\Delta f + b\nabla f - Vf$, $f \in C_0^\infty(\mathbb{R}^d)$, is a pre-generator for which we can prove its $L^\infty(\mathbb{R}^d, dx)$ -uniqueness. Moreover, we prove the $L^1(\mathbb{R}^d, dx)$ -uniqueness of weak solutions for the Fokker-Planck equation associated with this pre-generator.

Key Words: C_0 -semigroups; L^∞ -uniqueness; generalized Schrödinger operators; Fokker-Planck equation.

1 Preliminaries

Let E be a Polish space equipped with a σ -finite measure μ on its Borel σ -field \mathcal{B} . It is well known that, for a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ on $L^1(E, d\mu)$, its adjoint semigroup $\{T^*(t)\}_{t \geq 0}$ is no longer strongly continuous on the dual topological space $L^\infty(E, d\mu)$ of $L^1(E, d\mu)$ with respect to the strong dual topology of $L^\infty(E, d\mu)$. In [10] WU and ZHANG introduce on $L^\infty(E, d\mu)$ the topology of uniform convergence on compact subsets of $(L^1(E, d\mu), \|\cdot\|_1)$, denoted by $\mathcal{C}(L^\infty, L^1)$, for which the usual semigroups in the literature becomes C_0 -semigroups. If $\{T(t)\}_{t \geq 0}$ is a C_0 -semigroup on $L^1(E, \mu)$ with generator \mathcal{L} , then $\{T^*(t)\}_{t \geq 0}$ is a C_0 -semigroup on $(L^\infty(E, d\mu), \mathcal{C}(L^\infty, L^1))$ with generator \mathcal{L}^* . Moreover, one can prove that $(L^\infty(E, d\mu), \mathcal{C}(L^\infty, L^1))$ is complete and that the topological dual of $(L^\infty(E, d\mu), \mathcal{C}(L^\infty, L^1))$ is $(L^1(E, d\mu), \|\cdot\|_1)$. Let $\mathcal{A} : \mathcal{D} \longrightarrow L^\infty(E, d\mu)$ be a linear operator with its domain \mathcal{D} dense in $(L^\infty(E, d\mu), \mathcal{C}(L^\infty, L^1))$. \mathcal{A} is said to be a *pre-generator* in $(L^\infty(E, d\mu), \mathcal{C}(L^\infty, L^1))$, if there exists some C_0 -semigroup on $(L^\infty(E, d\mu), \mathcal{C}(L^\infty, L^1))$ such that its generator \mathcal{L} extends \mathcal{A} . We say that \mathcal{A} is an *essential generator* in $(L^\infty(E, d\mu), \mathcal{C}(L^\infty, L^1))$

*Institut Camille Jordan UMR 5208 (CNRS), Université Claude Bernard Lyon1, 69622 Villeurbanne, France and Engineering Faculty of Hunedoara, "Politehnica" University of Timișoara, 331128 Hunedoara, Romania e-mail: lemle_dan@yahoo.com

(or $(L^\infty(E, d\mu), \mathcal{C}(L^\infty, L^1))$ -unique), if \mathcal{A} is closable and its closure $\overline{\mathcal{A}}$ with respect to $\mathcal{C}(L^\infty, L^1)$ is the generator of some C_0 -semigroup on $(L^\infty(E, d\mu), \mathcal{C}(L^\infty, L^1))$. This uniqueness notion was studied in ARENDT [1], EBERLE [3], DJELLOUT [2], RÖCKNER [6], WU [8] and [9] and others in the Banach spaces setting and WU and ZHANG [10] and LEMLE [4] in the case of locally convex spaces.

2 L^∞ -uniqueness of generalized Schrödinger operators

In this note we consider the generalized Schrödinger operator

$$\mathcal{A}^V f := \frac{1}{2} \Delta f + b \nabla f - V f \quad , \quad \forall f \in C_0^\infty(\mathbb{R}^d) \quad (1)$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a measurable and locally bounded vector field and $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is a locally bounded potential. In the case where $V = 0$, the essential self-adjointness of $\mathcal{A} := \frac{1}{2} \Delta + b \nabla$ has been completely characterized in the works of WIELENS [7] and LISKEVITCH [5]. L^1 -uniqueness of this operator has been introduced and studied by WU [9], its L^p -uniqueness has been studied by EBERLE [3] and DJELLOUT [2] for $p \in [1, \infty)$ and by WU and ZHANG [10] for $p = \infty$.

Our purpose is to find some sufficient condition to assure the $L^\infty(\mathbb{R}^d, dx)$ -uniqueness of $(\mathcal{A}^V, C_0^\infty(\mathbb{R}^d))$ with respect to the topology $\mathcal{C}(L^\infty, L^1)$ in the case where $V \geq 0$.

At first, we must remark that the generalized Schrödinger operator $(\mathcal{A}^V, C_0^\infty(\mathbb{R}^d))$ is a pre-generator on $L^\infty(\mathbb{R}^d, dx)$. Indeed, if we consider the Feynman-Kac semigroup $\{P_t^V\}_{t \geq 0}$ given by

$$P_t^V f(x) := \mathbb{E}^x 1_{[t < \tau_e]} f(X_t) e^{-\int_0^t V(X_s) ds} \quad (2)$$

where $(X_t)_{0 \leq t < \tau_e}$ is the diffusion generated by \mathcal{A} and τ_e is the explosion time, then by [10, Theorem 1.4] it follows that $\{P_t^V\}_{t \geq 0}$ is a C_0 -semigroup on $L^\infty(\mathbb{R}^d, dx)$ with respect to the topology $\mathcal{C}(L^\infty, L^1)$. By Ito's formula one can prove that f belongs to the domain of the generator $\mathcal{L}_{(\infty)}^V$ of C_0 -semigroup $\{P_t^V\}_{t \geq 0}$ on $(L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))$.

The main result of this note is

Theorem 2.1. *Suppose that there is some measurable locally bounded function $\beta : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that*

$$\frac{b(x)x}{|x|} \geq \beta(|x|) \quad , \quad \forall x \in \mathbb{R}^d, x \neq 0. \quad (3)$$

Let $\tilde{\beta}(r) = \beta(r) + \frac{d-1}{2r}$. If the one-dimensional diffusion operator

$$\mathcal{A}_1^V = \frac{1}{2} \frac{d^2}{dr^2} + \tilde{\beta}(r) \frac{d}{dr} - V(r) \quad (4)$$

is $L^\infty(0, \infty; dx)$ -unique, then $(\mathcal{A}^V, C_0^\infty(\mathbb{R}^d))$ is $L^\infty(\mathbb{R}^d, dx)$ -unique. Moreover, for any $f \in L^1(\mathbb{R}^d, dx)$ the Fokker-Planck equation

$$\begin{cases} \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) - (\operatorname{div} b + V) u(t, x) \\ u(0, x) = f(x) \end{cases} \quad (5)$$

has one $L^1(\mathbb{R}^d, dx)$ -unique weak solution given by $u(t, x) = P_t^V f(x)$.

References

- [1] W. Arendt, The abstract Cauchy problem, special semigroups and perturbation, *Lect. Notes in Math.* **1184**(1986).
- [2] H. Djellout, Unicité dans L^p d'opérateurs de Nelson, Prépublication (1997).
- [3] A. Eberle, Uniqueness and non-uniqueness of singular diffusion operators, Doctor's thesis, University of Bielefeld (1997).
- [4] L.D. Lemle, Integrated semigroups of operators, uniqueness of pre-generators and applications, Doctor's thesis, Blaise Pascal University of Clermont-Ferrand (2007).
- [5] V. Liskevitch, On the uniqueness problem for Dirichlet operators, *J. Funct. Anal.* **162**(1999).
- [6] M. Röckner, L^p -analysis of finite and infinite dimensional diffusion operators, *Lect. Notes in Math.* **1715**(1998).
- [7] N. Wielens, On the essential self-adjointness of generalized Schrödinger operators, *J. Funct. Anal.* **61**(1985).
- [8] L. Wu, Uniqueness of Schrödinger Operators Restricted in a Domain, *J. Funct. Anal.* **153**(1998).
- [9] L. Wu, Uniqueness of Nelson's diffusions, *Probab. Theory Relat. Fields* **114**(1999).
- [10] L. Wu and Y. Zhang, A new topological approach for uniqueness of operators on L^∞ and L^1 -uniqueness of Fokker-Planck equations, *J. Funct. Anal.* **241**(2006).